Basic definitions in mathematical morphology

Dan Bloomberg

A binary image X is the set of *foreground* (FG) pixels in a two-dimensional array of pixels. Pixels not in the FG are in the *background* (BG). The pixels in X are defined in $\mathbb{Z}^2$ with respect to an *origin*, conventionally taken to be in the upper-left corner of the image.

A *structuring element* (SE) is a set B, also in $\mathbb{Z}^2$. In image processing, it is typically of smaller extent than the image X, and the location of its elements are defined with respect to an origin that is not necessarily contained within the elements of B.

The basic morphological operations are *erosion* and *dilation*. The erosion of a binary image X by a SE B is the set operation defined by

$$X \ominus B \ = \ \bigcap_{b \in B} X_{-b} \ = \ \{x \in \mathbb{Z}^2 \mid B_x \subseteq X\} \tag{1}$$

where X_{-b} is the *translation* of the image X by $-b$. The second definition states that erosion generates a set with a non-empty result at every location where the translate of B fits entirely within X. The dilation of an image X is defined by

$$X \oplus B \ = \ \bigcup_{b \in B} X_b \ = \ \bigcup_{x \in X} B_x \ = \ \{x + b \in \mathbb{Z}^2 \mid x \in X, b \in B\} \tag{2}$$

The first and second definitions state that dilation generates a set composed of the union of translations of X by elements in B, and *v.v.*. From the symmetry of these relations, it is clear that X and B commute under dilation

$$X \oplus B \ = B \oplus X \tag{3}$$

but they do not do so under erosion, as one might expect because addition is commutative but subtraction is not. (Historical note: $\ominus$ and $\oplus$ are *not* the original definitions of Minkowski subtraction and addition, respectively, which require an inversion of the SE about its origin.)

Erosion and dilation are *dual* operations:

$$(X \oplus B)^c \ = X^c \ominus \check{B} \tag{4}$$

where X^c, the *complement* of X, is the set of BG pixels of X, and $\check{B} = b| - b \in B$ is the *spatial inversion* of B through the origin.

The *opening* of an image X by a SE B is a composite operation of erosion followed by dilation:

$$X \circ B \ = (X \ominus B) \oplus B \tag{5}$$

Opening is *idempotent* and independent of the origin of the SE. It is also *anti-extensive*, because the result is guaranteed to be a subset of X. Opening can be visualized as the union of all translated SEs where the SE fits entirely within X.

The *closing* of an image X by a SE B is a composite operation of dilation followed by erosion:

$$X \bullet B \ = (X \oplus B) \ominus B \tag{6}$$

Closing, like opening, is idempotent and independent of the origin of the SE. It is *extensive*, because X is guaranteed to be a subset of the result.

Closing and opening are also dual operations:

$$(X \bullet B)^c \ = X^c \circ \check{B} \tag{7}$$

Using the dual relation, closing can be visualized as the complement of an opening of the background, i.e., as the complement of the union of all translations of the SE, such that the inverted SE fits entirely within X^c.

The *hit-miss transform* (HMT) is a morphological template matcher whose definition is based on the erosion operator. The HMT of a binary image X by a *disjoint* pair (τ_f, τ_b) of SEs is defined as the set transformation

$$X \otimes (\tau_f, \tau_b) \quad = \quad (X \ominus \tau_f) \ \cap \ (X^c \ominus \tau_b) \tag{8}$$

The HMT generates a set with non-empty result at every location where both the FG SE τ_f fits entirely within X and the BG SE τ_b fits entirely within X^c. It is common to speak of the elements in τ_f as *hits*, of elements in τ_b as *misses*, and elements not in their union as *don't-cares*.